Wilson loops in Five-Dimensional Super-Yang-Mills

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Abstract

We consider circular non-BPS Maldacena-Wilson loops in five-dimensional supersymmetric Yang-Mills theory ($d = 5$ SYM) both as macroscopic strings in the D4-brane geometry and directly in gauge theory. We find that in the Dp-brane geometries for increasing p, $p = 4$ is the last value for which the radius of the string worldsheet describing the Wilson loop is independent of the UV cut-off. It is also the last value for which the area of the worldsheet can be (at least partially) regularized by the standard Legendre transformation. The asymptotics of the string worldsheet allow the remaining divergence in the regularized area to be determined, and it is found to be logarithmic in the UV cut-off. We also consider the M2-brane in $AdS_7 \times S^4$ which is the M-theory lift of the Wilson loop, and dual to a "Wilson surface" in the $(2, 0)$, $d = 6$ CFT. We find that it has exactly the same logarithmic divergence in its regularized action. The origin of the divergence has been previously understood in terms of a conformal anomaly for surface observables in the CFT. Turning to the gauge theory, a similar picture is found in $d = 5$ SYM. The divergence and its coefficient can be recovered for general smooth loops by summing the leading divergences in the analytic continuation of dimensional regularization of planar rainbow/ladder diagrams. These diagrams are finite in $5 - \epsilon$ dimensions. The interpretation is that the Wilson loop is renormalized by a factor containing this leading divergence of six-dimensional origin, and also subleading divergences, and that the remaining part of the Wilson loop expectation value is a finite, scheme-dependent quantity. We substantiate this claim by showing that the interacting diagrams at one loop are finite in our regularization scheme in $d = 5$ dimensions, but not for $d \geq 6$.

1 Introduction, main results, and conclusions

The Maldacena-Wilson loop [1, 2] continues to be a remarkably useful observable in the context of the gauge-gravity duality. In the context of AdS/CFT, the circular Wilson loops [3–8] have proven to be amenable to exact calculation using the techniques of localization [9–12], providing hard predictions for a range of stringy and M-theoretic phenomena including semi-classical fundamental strings and membranes, D-branes, and bubbling geometries [13–21]. Continuing these successes outside the regime of conformal symmetry, in particular to maximally supersymmetric Yang-Mills (SYM) theories in dimensions other than four, is an important step towards understanding the gauge-gravity duality in these far-less-explored contexts.

In this paper we will consider the circular Maldacena-Wilson loop with constant scalar coupling (see (32) for a definition) in $d = 5$ SYM. In this five-dimensional context the circular Wilson loop preserves no global supersymmetries. The first and most obvious question is whether the $d = 5$ theory is sensible, since by standard power-counting it is a non-renormalizable theory. We will follow the procedure of using dimensional reduction from $\mathcal{N} = 1$, $d = 10$ SYM to 2ω dimensions. Since we are *above* rather than below four dimensions (where dimensional regularization actually renders integrals over loop momenta UV-finite), this procedure is viewed as an analytic continuation from convergent results at $d < 4$ to $d = 5$. Despite this questionable regularization scheme, we find that we can make contact with the string dual, i.e. a fundamental string in the D4-brane geometry [22], and its M2-brane lift.

It appears that in the Dp-brane geometries, for increasing p, $p = 4$ (corresponding to $d = 5$ SYM) is in some sense a final outpost. In the work [23], the embedding functions for strings dual to 1/4 BPS circular Wilson loops with trivial expectation value (a generalization of the Zarembo loops [24]) in SYM for $2 \leq d \leq 8$ were presented. There is a stark division in the behaviour of the worldsheets as they approach the boundary of the geometry precisely between $p = 4$ and $p \geq 5$. Specifically, for $p \leq 4$, the radius R of the worldsheet (dual to the radius of the Wilson loop contour), assumes a constant value as the boundary is approached. For $p > 4$ this is no longer true and a UV cut-off must explicitly be added in order to define R – or equivalently – R becomes a function of the UV cut-off, see figure 1, taken from [23]. The figure shows a plot of the string profile with the holographic direction plotted on the vertical axis, and a boundary radial direction plotted on the horizontal axis. In section 2.1 we review the details of these solutions.

It turns out that this behaviour is generic, in the sense that it persists for the duals of non-BPS circular Wilson loops, which unlike the 1/4 BPS ones, have a constant coupling to the scalar fields of SYM. In section 2.2, we carry out the analysis of the string worldsheets for these Wilson loops. Although we are unable to find exact solutions for the embeddings, we can find their asymptotic behaviour as the boundary is approached. Using this information we show that R is independent of the UV cut-off only for $p < 5$, as for the BPS solutions.

The area of the worldsheets describing the Wilson loops is infinite for any p. In the $AdS_5 \times S^5$ case (i.e. $p = 3$), this divergence is well understood and is removed by a Legendre transformation [25]. In the case of the 1/4 BPS loops in the Dp-brane
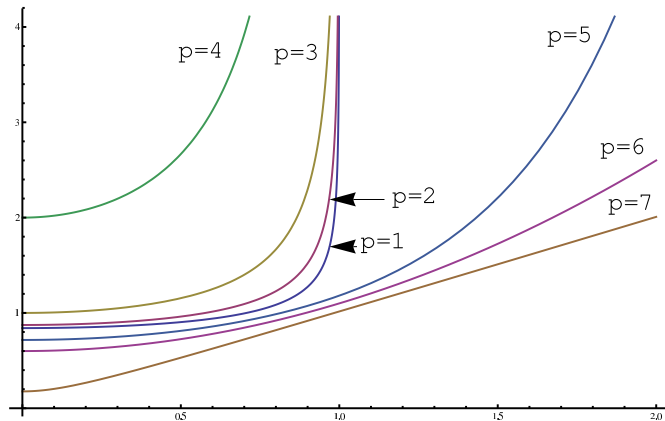
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Figure 1: Profiles of string duals of 1/4 BPS circular Wilson loops (6) in the Dp-brane geometries (5). The holographic U direction is the vertical axis, the boundary radial coordinate r is the horizontal axis. There is a marked difference between $p > 4$ and $p \leq 4$. Taken from [23].

geometries, the same Legendre transformation simply eliminates the area entirely, giving the trivial expectation value for the Wilson loop $e^0 = 1$. For the non-BPS Wilson loops, we find that again $p = 4$ is a special value. It is the last value for which the Legendre transformation removes the (in this case leading) divergence of the area, leaving a log(UV cut-off) divergence and a finite piece. The asymptotics of the string worldsheet allow the determination of the coefficient of the log, while the finite piece is not obtainable. It is, in any case not well defined, since a shift in the UV cut-off would affect it. The details of these calculations are given section 2.2. We quote the result here for convenience[1]

$$\langle W \rangle = (\text{prefactor}) \cdot \exp\left(-S_{\text{reg.}}\right) = (\text{prefactor}) \cdot \exp\left(\frac{g^2 N}{16\pi R} \log U_{\text{max.}}\right) \cdot (\text{finite}), \quad (1)$$

where $U_{\text{max.}}$ is the cut-off in the holographic direction, see (5), and where "prefactor" refers to the semi-classical dressing of the main exponential portion of the partition function, see [23] for a discussion.

The fact that the $p = 4$ case shares these two "nice" features with its lower dimensional cousins, i.e. regularizable worldsheet area and radius independent of the cut-off, certainly resonates with recent speculations about the possible finiteness of $d = 5$ SYM [26, 27]. We will see that, using dimensional regularization analytically continued to five-dimensions, the gauge theory makes contact with this behaviour of the string dual. Specifically, we perform an analysis of the planar rainbow/ladder diagrams in section 3. The "loop-to-loop" propagator $P(\tau_1, \tau_2)$ (34) has the following

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[1]Note that U has dimensions of energy. There is of course a scale (not shown) giving a dimensionless argument for the logarithm. It is set by the minimum value of U plumbed by the worldsheet, which is in turn related to the radius of the loop.

behaviour

$$P(\tau_1, \tau_2) \propto \frac{1}{\sin^{2\omega-4}\frac{\tau_{12}}{2}}, \tag{2}$$

where $\tau_{12} = \tau_1 - \tau_2$ is the difference between the Wilson loop contour parameter at the two points where the propagator is joined, and the dimension $d = 2\omega$. We find that a certain sub-class of planar rainbow/ladder diagrams contributes the highest divergence, order-by-order in the perturbative expansion. This allows us to sum-up all of these contributions, finding

$$\langle W \rangle_{\text{perturbative}} = (\text{prefactor}) \cdot \exp\left(\frac{g^2 N}{16\pi R}\frac{1}{\epsilon}\right) \cdot (\text{finite}), \tag{3}$$

where $\epsilon = 5 - 2\omega$. Equating $\log U_{\text{max.}}$ with $1/\epsilon$ we find an exact match. In fact if we set $2\omega = 5$ and use instead a point-splitting regularization δ of the Wilson loop contour, we obtain $-\log\delta$ in place of $1/\epsilon$, see section 3. We will suggest that the subleading divergences should be associated with the prefactor of the exponential term in the Wilson loop VEV.

Note that for $d = 2\omega$ slightly below 5, the loop-to-loop propagator integrates to a finite quantity. This appears to be a gauge theory reflection of the fact that $d = 5$ is the last sensible setting for the Wilson loop in the string dual described above. We might then expect that the interacting diagrams are at least subleading, if not finite. Evaluating the $\mathcal{O}(g^4)$ corrections, in section 3.2, we find that they are indeed finite when 2ω is set (i.e. analytically continued) to 5, but not for $2\omega \geq 6$. Again, we see that the gauge theory picture is sensible (at least in our regularization scheme), like in the string case, for the last time at $d = 5$, as the dimension is increased.

The D4-brane geometry cannot be trusted at arbitrarily close distances to the boundary, and at a certain point the M-theory description takes over[2] and the background geometry becomes $AdS_7 \times S^4$. The lift of the non-BPS circular Wilson loop is an M2-brane and is considered in section 2.2.2. The action of this M2-brane may be regularized via an analogous Legendre transformation and has precisely the same logarithmic term as the string in the D4-brane geometry. This logarithmic divergence may be viewed as a conformal anomaly of the dual surface operator in the $d = 6$ $(2, 0)$ CFT [28–31]. In section 2.2.3 we show that the regularized area of the string worldsheet in the D4-brane geometry reproduces this anomaly term for general-shaped smooth and closed contours; in section 3.3 the corresponding generalization in the gauge theory is performed. It is remarkable that we have recovered this anomaly directly in $d = 5$ SYM, especially given recent speculations that the $d = 6$ CFT might be captured entirely by $d = 5$ SYM [26, 27, 32–34].

There are various extensions of the present work which could be considered. There is a kind of conformal symmetry at play in the Dp-brane backgrounds and the associated SYM theories [35]. It would be interesting to understand whether this symmetry can be used to obtain the circular Wilson loop considered here as a transformation of the 1/2 BPS straight line [23] (which has trivial expectation value), as is the case in $d = 4$ [4]. This may provide an alternate derivation of the leading divergence in

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[2]This happens for $g^2 U \sim N^{1/3}$.

terms of a (generalized) conformal anomaly (in this case directly in $d = 5$ SYM). The exponential factor dressing the Wilson loop expectation value is very reminiscent of Wilson loop renormalization in four dimensions [36–39]. This suggests that there is perhaps a way to give physical meaning to the finite part of the expectation value, through a subtraction scheme. It would also be interesting to consider correlators with local operators. This calculation has been carried out for the spherical Wilson surface in $AdS_7 \times S^4$ in [40], where, unlike for the expectation value of the surface itself, finite results are obtained. Here one would require the explicit solution for the M2-brane describing the Wilson loop; the techniques for computing holographic correlation functions in the Dp-brane backgrounds are also available [41, 42]. It is also interesting to ask to what extent contact can be made with the 1/2 BPS circular Wilson loop in $\mathcal{N} = 4$, $d = 4$ SYM through dimensional reduction, especially as regards the S-duality of the latter as discussed in [27], and whether localization techniques have any application to the calculation of the five-dimensional Wilson loop expectation value. Finally, we note that we have not considered non-perturbative corrections to the Wilson loop. It would be very interesting to explore their effect on the expectation value.

2 String duals of BPS and non-BPS Wilson loops

In this section we will look at the generalization of the circular Zarembo loops studied in [23], remarking that the string duals require a cut-off in order to be defined in the Dp-brane backgrounds for $p > 4$. We will then continue with a generalization of these arguments to regular circular Wilson loops, which, apart from the conformal case $p = 3$, are non-BPS.

2.1 1/4 BPS Wilson loops

In the work [23], string solutions were found which are dual to Maldacena-Wilson loops in d-dimensional SYM with circular contours and with scalar couplings which also describe a (great) circle

$$x^\mu = R\left(\cos\tau,\, \sin\tau,\, 0, \ldots, 0\right), \qquad \Theta^I = (-\sin\tau,\, \cos\tau,\, 0, \ldots, 0). \tag{4}$$

The string duals are fundamental strings in the Dp-brane geometries [22][3]

$$ds^2 = \alpha'\left(\frac{U^{(7-p)/2}}{C_p}dx_{\shortparallel}^2 + \frac{C_p}{U^{(7-p)/2}}\,dU^2 + C_p\,U^{(p-3)/2}\,d\Omega_{8-p}^2\right),$$

$$e^\phi = (2\pi)^{1-p}g^2\left(\frac{C_p^2}{U^{7-p}}\right)^{(3-p)/4}, \quad C_p^2 = g^2 N\,2^{6-2p}\pi^{(9-3p)/2}\,\Gamma\left(\frac{7-p}{2}\right). \tag{5}$$

We let $dx_{\shortparallel}^2 = dr^2 + r^2 d\varphi^2 + dx_{\mathbb{R}^{p-1}}^2$, and $d\Omega_{8-p} = d\theta^2 + \cos^2\theta\,d\phi^2 + \sin^2\theta\,d\Omega_{6-p}^2$. We then set $\varphi = \phi$ to one of our string worldsheet coordinates, and let r, U, and θ depend

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[3]The definition of C_p and the dilaton are consistent with the normalization of the SYM action $S = \frac{1}{g^2}\int d^{p+1}x\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \ldots$.

only on the other one. Then the string solutions are expressed as (for $p \leq 7$)

$$r(U) = \begin{cases} \sqrt{\frac{2\,C_p^2}{5-p}}\sqrt{U_{\mathrm{min.}}^{p-5} - U^{p-5}}, & p < 5 \\ \sqrt{\frac{2\,C_p^2}{p-5}}\sqrt{U^{p-5} - U_{\mathrm{min.}}^{p-5}}, & p > 5 \,, \\ \sqrt{2\,C_5^2\,\log\frac{U}{U_{\mathrm{min.}}}}, & p = 5 \end{cases} \qquad \sin\theta = \frac{U_{\mathrm{min.}}}{U}, \qquad (6)$$

and describe a string which wraps a hemisphere in the S^{8-p} and ends along a circular contour of radius R at the boundary ($U = U_{\mathrm{max.}} \gg 1$) of the remainder of the geometry. The worldsheet smoothly closes-off at $U = U_{\mathrm{min.}}$ where $r(U) = 0$.

We now remark that in the cases where $p \geq 5$, the radius of the Wilson loop depends upon the cut-off $U_{\mathrm{max.}}$.

$$R = \begin{cases} \sqrt{\frac{2\,C_p^2}{5-p}}\sqrt{U_{\mathrm{min.}}^{p-5}}, & p < 5 \\ \sqrt{\frac{2\,C_p^2}{p-5}}\sqrt{U_{\mathrm{max.}}^{p-5} - U_{\mathrm{min.}}^{p-5}}, & p > 5 \,. \\ \sqrt{2\,C_5^2\,\log\frac{U_{\mathrm{max.}}}{U_{\mathrm{min.}}}}, & p = 5 \end{cases} \qquad (7)$$

Thus the Wilson loops may be defined in the gravity duals to SYM in $d \leq 5$ dimensions without recourse to a UV cut-off; the same is not true for $d > 5$.

The 1/4 BPS Wilson loops considered in this section are rather special objects which have trivial (unit) expectation value[4]. In section 2.2 we will consider non-BPS circles for which this is not the case.

2.2 Non-BPS Wilson loops

In section 2.1 we showed that certain 1/4 BPS circular Wilson loops in d-dimensional SYM had string duals which could be defined without recourse to a UV cut-off if $d = p + 1 \leq 5$. Here we will consider a circular Wilson loop with a constant coupling to the scalars, so that the string dual sits at a point on the S^{8-p}. For $p \neq 3$ this object preserves no global supersymmetries. We will not be able to solve for the string embedding, but we will derive its asymptotic form as the boundary is approached. This we will use to analyze the leading divergences in the worldsheet area.

We consider a fundamental string ending in a circular contour on the boundary of the Dp-brane geometry (5). We let $dx_{\parallel}^2 = dr^2 + r^2 d\varphi^2 + dx_{\mathbb{R}^{p-1}}^2$, and take our string worldsheet to be parameterized by U and φ. Using the ansatz whereby $r(U)$, the Nambu-Goto action is

$$S = \int dU\, r\, \sqrt{1 + \frac{U^{7-p}}{C_p^2} r'^2}, \qquad (8)$$

where $f' \equiv \partial_U f$ and we have integrated over φ since the Lagrangian is independent of it. One can then easily verify that for $p \leq 4$, near the boundary at $U = \infty$

$$r(U) = R - \frac{1}{5-p}\frac{C_p^2}{R}\frac{1}{U^{5-p}} + \ldots \qquad (9)$$

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[4]This is because the Legendre transformation (12) exactly cancels the bare action [23].

is a solution to the equation of motion. In the full solution, the worldsheet closes-off at some minimum value of U, $U_{\text{min.}}$. The radius R is then related to $U_{\text{min.}}$, in much the same way as the solutions presented in section 2.1. As we saw in that section, here there is a similar marked difference for $p > 4$. When we take $p > 4$ we find that $r(U)$ diverges as $U \to \infty$, requiring the radius of the Wilson loop to be defined at some cut-off $U_{\text{max.}}$. For example for $p = 5$ one finds[5]

$$r(U) = R - \frac{C_5^2}{R} \log \frac{U_{\text{max.}}}{U} + \dots . \tag{10}$$

In this situation, as in section 2.1, R is a function of *both* $U_{\text{min.}}$ *and* $U_{\text{max.}}$. Thus the radius of the Wilson loop is affected by changes to the cut-off. Given the apparent non-renormalizability of supersymmetric Yang-Mills in $d > 4$ dimensions, one would have expected this behaviour to set-in already at $p = 4$. The fact that it is postponed to $p \geq 5$ is interesting, given recent speculations that SYM in $d = 5$ may be a finite theory.

2.2.1 Regularized area of the worldsheet

We would now like to analyze the divergence in the area of the worldsheet corresponding to the non-BPS circular Wilson loops. The area of a Wilson loop is regularized via a Legendre transformation using the $\mathcal{Y}$ coordinates defined as

$$\frac{dU^2}{U^2} + d\Omega_{8-p}^2 = \frac{d\mathcal{Y}^I d\mathcal{Y}^I}{\mathcal{Y}^2}, \quad \mathcal{Y}^I = U\hat{\theta}^I, \quad \hat{\theta}^I \hat{\theta}^I = 1, \quad I = 1, \dots, 9 - p. \tag{11}$$

Then the regularized area of the worldsheet Σ is defined as [25]

$$
\begin{aligned}
S_{\text{reg.}} &= S - \int d\tau \, d\sigma \, \partial_\sigma \left(\mathcal{Y}^I \frac{\delta \mathcal{L}}{\delta \partial_\sigma \mathcal{Y}^I} \right) \\
&= S - \int d\tau \, \mathcal{Y}^I \frac{\delta \mathcal{L}}{\delta \partial_\sigma \mathcal{Y}^I} \bigg|_{\partial \Sigma} ,
\end{aligned}
\tag{12}
$$

where the τ coordinate parametrizes the boundary contour, and $\mathcal{L}$ indicates the Lagrangian density, so that $S = \int d\tau d\sigma \mathcal{L}$. We then find that

$$S_{\text{reg.}} = \int_{U_{\text{min.}}}^{U_{\text{max.}}} dU \, r \sqrt{1 + \frac{U^{7-p}}{C_p^2} r'^2} - \frac{U \, r}{\sqrt{1 + \frac{U^{7-p}}{C_p^2} r'^2}} \bigg|_{U=U_{\text{max.}}} . \tag{13}$$

Specializing to $p = 4$ we may calculate the regularized area of the worldsheet using (9). One finds[6]

$$
\begin{aligned}
S &= \int^{U_{\text{max.}}} dU \left(R - \frac{C_4^2}{2RU} + \dots \right) = R \, U_{\text{max.}} - \frac{C_4^2}{2R} \log U_{\text{max.}} + \text{finite}, \\
S_{\text{reg.}} &= -\frac{C_4^2}{2R} \log U_{\text{max.}} + \text{finite}.
\end{aligned}
\tag{14}
$$

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[5]Similar behaviour, i.e. $r(U)$ diverging as $U \to \infty$ is found for $p > 5$.

[6]The boundary term in the Legendre transformation also contributes to the finite piece.

The situation is vastly different for $p > 4$. For example, for $p = 5$, using (10), we find that[7]

$$S = \sqrt{C_5^2 + R^2}\, U_{\text{max.}} + \text{finite}, \qquad \left.\frac{U\,r}{\sqrt{1 + \frac{U^{7-p}}{C_p^2}r'^2}}\right|_{U=U_{\text{max.}}} = \frac{R^2}{\sqrt{C_5^2 + R^2}}\, U_{\text{max.}}, \quad (15)$$

and therefore the regularization procedure does not remove the leading divergence[8]. This may be an indication that SYM in dimensions only greater than five are non-renormalizable. Using (14), the Wilson loop expectation value for $p = 4$ is given by

$$\langle W_{\text{circle}} \rangle = \mathcal{V}\, e^{-S_{\text{reg.}}} = \mathcal{V} \exp\left(\frac{g^2 N}{16\pi R} \log U_{\text{max.}}\right) \cdot (\text{finite}), \quad (16)$$

where we have indicated the appearance of an unknown, and we will argue from gauge theory, also divergent prefactor $\mathcal{V}$, which can in principle be determined using semi-classical methods.

In section 3 we will recover the exponential factor in (16) by summing planar rainbow/ladder diagrams, and posit that the remaining finite factor is provided by interacting diagrams. Before doing so we would like to consider the uplift of the $p = 4$ case to M-theory, where we will see a six-dimensional origin of the exponential factor.

2.2.2 M-theory lift

For strong coupling, defined as $g^2 U \gg N^{1/3}$, the IIA D4-brane geometry is replaced by the M-theory background $AdS_7 \times S^4$ with a boundary direction x_6 periodicially identified on a circle of radius $R_6 = g^2/(8\pi^2)$ [22]. The metric on this space may be expressed as[9]

$$ds^2 = 4(\pi N)^{2/3} l_p^2 \left(\frac{d\tilde{U}^2}{\tilde{U}^2} + \tilde{U}^2 \left(dr^2 + r^2 d\phi^2 + dx_6^2 + dx_a^2\right) + \frac{1}{4}d\Omega_4^2\right), \quad (17)$$

where $a = 1, \ldots, 3$. We then consider an M2-brane with worldvolume coordinates $\{\tilde{U}, \phi, x_6\}$ (i.e. wrapped on x_6) and take $r(\tilde{U})$. Shrinking the M-theory circle x_6 to zero size, we recover the IIA fundamental string describing the circular Wilson loop in the D4-brane geometry. The boundary surface of the M2-brane is $S^1 \times S^1$, i.e. the Wilson loop circle times the x_6 circle. Integrating over x_6 and ϕ we obtain

$$S_{M2} = 8\pi N R_6 \int d\tilde{U}\, r\, \tilde{U}\, \sqrt{1 + \tilde{U}^4 r'^2}, \quad (18)$$

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[7]The conditions under which a Legendre transform can remove the leading divergence has been analyzed in great detail in [43]. The results given here are a special case of that analysis.

[8]Again, the same general behaviour, i.e. leading divergences not removed by the Legendre transformation, is found for $p > 5$.

[9]The coordinate $\tilde{U}$ is related to the U coordinate of the D4-brane geometry via $\tilde{U}^2 = 2\pi U/(g^2 N)$, see [22].

where we have used the M2-brane tension $T = l_p^{-3}/(2\pi)^2$. We then find that the equation of motion for large $\tilde{U}$ is solved by

$$r(\tilde{U}) = R - \frac{1}{4\tilde{U}^2 R} + \dots. \tag{19}$$

The action of the M2-brane then evaluates to

$$\frac{S_{M2}}{8\pi N R_6} = \int^{\tilde{U}_{\text{max.}}} d\tilde{U} \left(R\tilde{U} - \frac{1}{8\,R\tilde{U}} + \dots \right) = \frac{1}{2} R\tilde{U}_{\text{max.}}^2 - \frac{1}{8R} \log \tilde{U}_{\text{max.}} + \text{finite.} \tag{20}$$

The Legendre transformation may be implemented in analogy with the fundamental string case. One defines variables Y^I such that

$$\frac{d\tilde{U}^2}{\tilde{U}^2} + \frac{1}{4} d\Omega_4^2 = \frac{1}{4} \left(\frac{dV^2}{V^2} + d\Omega_4^2 \right) = \frac{dY^I dY^I}{4Y^2},$$
$$Y^I = V\hat{\theta}^I, \quad \hat{\theta}^I \hat{\theta}^I = 1, \quad I = 1, \dots, 5, \tag{21}$$

so that $V = \tilde{U}^2$. Then the action is regularized via

$$S_{M2\,\text{reg.}} = S_{M2} - \int d^2\tau \, d\sigma \, \partial_\sigma \left(Y^I \frac{\delta\mathcal{L}}{\delta\partial_\sigma Y^I} \right)$$
$$= S_{M2} - \int d^2\tau \, Y^I \frac{\delta\mathcal{L}}{\delta\partial_\sigma Y^I} \bigg|_{\partial\Sigma}, \tag{22}$$

where the boundary surface $\partial\Sigma$ is parameterized by the two τ coordinates. This removes the leading divergence[10] from (20). We then find that the expectation value of our Wilson surface is

$$\langle W_{\text{surface}} \rangle = \tilde{\mathcal{V}} \exp\left(-S_{M2\,\text{reg.}}\right) = \tilde{\mathcal{V}} \exp\left(\frac{\pi N R_6}{R} \log \tilde{U}_{\text{max.}} \right) \cdot (\text{finite}), \tag{23}$$

where we have included an unknown semi-classical prefactor $\tilde{\mathcal{V}}$ as in the string case. Using the fact that $\tilde{U} \sim U^{1/2}$ (see footnote 9), and the identification $g^2/(8\pi^2) = R_6$, the string (16) and M-theory (23) results match, at least for the exponential factor containing the logarithmic divergence and for the finite part, since the action (18) is equivalent to the string action (8) with $p = 4$. The prefactor is sensitive to fluctuations of the classical solutions and could be different between M and string theory.

A similar logarithmic divergence is seen in the spherical M2-brane solution in $AdS_7 \times S^4$ presented in [44] (section 5). The metric on AdS_7 is taken as $dU^2/U^2 + U^2 dx_i^2$ and the solution is

$$x^i(U, \theta, \phi) = \sqrt{R^2 - U^{-2}} \left(\sin\theta\cos\phi, \, \sin\theta\sin\phi, \, \cos\theta, 0, 0, 0 \right). \tag{24}$$

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[10] As in the string case the boundary term in the Legendre transformation also contributes a finite term.

The M2-brane action then evaluates to

$$S_{M2} = 4N \left(R^2 U_{\text{max.}}^2 - \log\left(2\,R\,U_{\text{max.}}\right) - \frac{1}{2} \right). \tag{25}$$

Again, the regularization procedure (22) removes the leading term. Then we are left with a logarithmic divergence, causing the expectation value to be scale-dependent. Since M-theory on $AdS_7 \times S^4$ is dual to the $(2,0)$ $d = 6$ CFT, this scale dependence might appear surprising. As mentioned in the introduction, it has been understood as a conformal anomaly suffered by sub-manifold observables corresponding to k-branes for even k (such as Wilson surfaces) in CFT's [28–31]. We may therefore interpret the logarithmic divergence in (16) as arising in $d = 5$ SYM, via dimensional reduction, from this $d = 6$ anomaly. It is remarkable that, given the recent speculations that the $(2,0)$ CFT in six dimensions might actually also be described by five-dimensional SYM [26, 27, 32–34], we are able to recover this exponential term by summing planar ladder diagrams directly in $d = 5$ SYM, see section 3.

2.2.3 Generalization to smooth, closed contours

Before turning to the gauge theory analysis, we remark that the coefficient of the log divergence is known for general shaped Wilson surfaces. It is given by the rigid string action [28, 44]

$$S_{\text{rigid}} = \frac{N}{4\pi} \int d^2\tau \,\sqrt{h} \left(\nabla^2 X^i\right)^2, \tag{26}$$

where the embedding of the surface in $\mathbb{R}^6$ is given by $X^i(\tau_1, \tau_2)$ and where $h_{\alpha\beta} = \partial_\alpha X^i \partial_\beta X^i$ is the induced metric. If we take τ_1 to parametrize the 5-d Wilson loop contour $x^I(\tau_1)$ by arclength, and τ_2 to be the trvially embedded periodically identified M-theory direction,

$$X^i(\tau_1, \tau_2) = \left(x^I(\tau_1), \tau_2\right), \qquad I = 1, \ldots, 5, \tag{27}$$

this reduces to

$$S_{\text{rigid}} = \frac{N R_6}{2} \int d\tau_1 \left(\partial_{\tau_1}^2 x^I\right)^2. \tag{28}$$

We may now consider a string worldsheet in the $D4$-brane geometry with embedding functions $x^I(\tau_1, U)$ such that $x^I(\tau_1, \infty) = x^I(\tau_1)$, i.e. a string describing the Wilson loop. The string action reads

$$S = \frac{1}{2\pi} \int d\tau_1\, dU \sqrt{\dot{x}^2 \left(1 + x'^2 \frac{U^3}{C_4^2}\right) - (\dot{x} \cdot x')^2 \frac{U^3}{C_4^2}}, \tag{29}$$

where we take $\dot{x}^I \equiv \partial_{\tau_1} x^I$ and $x'^I \equiv \partial_U x^I$. One may verify that for large U, the following ansatz satisfies the equations of motion

$$x^I(\tau_1, U) \simeq x^I(\tau_1) + \frac{C_4^2}{U} \ddot{x}^I(\tau_1), \tag{30}$$

9

where we remind the reader that $x^I(\tau_1)$ is parametrized by arclength. Evaluating the action for large U one obtains

$$S = \frac{1}{2\pi} L\, U_{\text{max.}} - \frac{g^2 N}{32\pi^2}\left(\int d\tau_1\, \ddot{x}^2\right)\log U_{\text{max.}} + \text{subleading}, \qquad (31)$$

where L is the arclength of the curve and where we have used the fact that $C_4^2 = g^2 N/(8\pi)$. The Legendre transformation removes the leading divergence. Then, using the dictionary between M-theoretic and string theoretic quantities as in section 2.2.2, we find that we have an exact match with (28).

3 Gauge theory analysis

3.1 Exponential factor from planar diagrams

In this section we will recover (16) by summing ladder diagrams in Euclidean five-dimensional SYM, finding the exact exponent. The Wilson loop is defined in the gauge theory as follows

$$W = \frac{1}{N}\text{Tr}\mathcal{P}\exp\oint d\tau\left(i\dot{x}^\mu A_\mu + |\dot{x}|\Theta^I\Phi_I\right), \qquad (32)$$

where Φ_I with $I = 1,\ldots,5$ are the five real scalars of the SYM theory, A_μ is the gauge field, and $\mathcal{P}$ indicates path ordering. The trace will be taken in the fundamental representation of the gauge group $SU(N)$, whose generators T^a are normalized by $\text{Tr}(T^a T^b) = \delta^{ab}/2$.

We take $x^\mu = R\left(\cos\tau, \sin\tau, 0, 0, 0\right)$ and $\Theta^I = \text{const.}$. Using dimensional reduction from $\mathcal{N} = 1$, $d = 10$ SYM to $d = 2\omega$ dimensions [3], the Feynman-gauge propagators are as follows

$$\langle A_\mu^a(x)\, A_\nu^b(0)\rangle = \Gamma(\omega - 1)\frac{g^2}{4\pi^\omega}\frac{\delta^{ab}\delta_{\mu\nu}}{x^{2\omega-2}}, \quad \langle \Phi_I^a(x)\, \Phi_J^b(0)\rangle = \Gamma(\omega - 1)\frac{g^2}{4\pi^\omega}\frac{\delta^{ab}\delta_{IJ}}{x^{2\omega-2}}. \quad (33)$$

The fundamental object of interest is the loop-to-loop propagator which refers to the 1-gluon + 1-scalar exchange between two locations $x_1^\mu = x^\mu(\tau_1)$ and $x_2^\mu = x^\mu(\tau_2)$ on the Wilson loop contour

$$\left\langle \left(i\dot{x}^\mu A_\mu^a + |\dot{x}|\Theta^I\Phi_I^a\right)(x_1)\left(i\dot{x}^\mu A_\mu^b + |\dot{x}|\Theta^I\Phi_I^b\right)(x_2)\right\rangle = \frac{g^2\Gamma(\omega-1)}{\pi^\omega R^{2\omega-4}}\frac{\delta_{ab}\,2^{1-2\omega}}{\sin^{2\omega-4}\frac{\tau_1-\tau_2}{2}}. \quad (34)$$

We see that an integral over τ_1 and τ_2 will produce a $1/(5-2\omega) \equiv 1/\epsilon$ divergence as τ_2 approaches τ_1. Alternatively, we could set $2\omega = 5$ and regulate this divergence using point-splitting, i.e. by cutting the integral off at $\tau_1 - \tau_2 = \delta$. We will continue by considering both regularizations.

Summing the planar rainbow/ladders was done for the $d = 4$ theory in [3], the difference here is that our loop-to-loop propagator is not constant. For this reason we will need to take a closer look at the diagrams. It was shown in [3], that the number
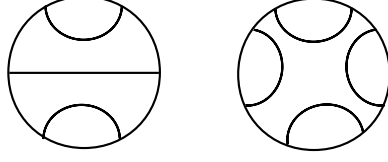
Figure 2: Examples of (q, k)-graphs. On the left a $(2, 3)$-graph is shown; on the right a $(4, 4)$-graph is shown.

N_k of planar rainbow/ladder diagrams with k propagators is given by the kth Catalan number

$$N_k = C_k = \frac{(2k)!}{k!(k+1)!}. \tag{35}$$

It will be important for our considerations to further subdivide the diagrams by the number of outermost propagators they contain, see figure 3.1. An outermost propagator is defined as a propagator which encloses, between itself and the Wilson loop contour, no other propagators. Let us denote the diagrams with q outermost propagators and k total propagators as (q, k)-graphs. It is obvious that $q \in [2, k]$. The multiplicity $M_{q,k}$ of the (q, k)-graphs is[11]

$$M_{q,k} = \frac{2\, k!(k-2)!}{q!(q-2)!(k-q)!(k-q+1)!}, \tag{36}$$

and one can verify that

$$\sum_{q=2}^{k} M_{q,k} = N_k, \tag{37}$$

as it must.

One finds that there is an association, at a given loop-level (i.e. fixed k), between the divergence of a graph and its q-value, the maximum divergence coming from the maximum value of q, i.e. $q = k$. Indeed we find that

$$(q, k)\text{-graph} \propto \begin{cases} (\frac{1}{\epsilon})^q + \text{subleading}, & \text{dim. red.} \\ (\log \delta)^q + \text{subleading}, & \text{point-split.} \end{cases} \tag{38}$$

We would therefore like to sum-up the most divergent diagrams, the (k, k)-graphs. It turns-out that the integration associated to these graphs has a simple closed form at the leading order in small-ϵ. The integral is as follows

$$I_{k,k} = \frac{2\pi}{2k} \int_0^{2\pi} d\theta_{2k-1} \int_0^{\theta_{2k-1}} d\theta_{2k-2} \ldots \int_0^{\theta_2} d\theta_1 \, \frac{1}{\sin^{1-\epsilon} \frac{\theta_1}{2}} \prod_{j=2}^{k} \frac{1}{\sin^{1-\epsilon} \frac{\theta_{2j-1}-\theta_{2j-2}}{2}}, \tag{39}$$

---

[11]There is only one $(1, 1)$-graph – the only graph with one propagator – and so the formula applies only to diagrams with two or more propagators.

in dimensional reduction, or

$$I_{k,k} = \frac{2\pi}{2k} \int_{\delta}^{2\pi-\delta} d\theta_{2k-1} \int_{\delta}^{\theta_{2k-1}-\delta} d\theta_{2k-2} \dots \int_{\delta}^{\theta_2-\delta} d\theta_1 \frac{1}{\sin\frac{\theta_1}{2}} \prod_{j=2}^{k} \frac{1}{\sin\frac{\theta_{2j-1}-\theta_{2j-2}}{2}}, \quad (40)$$

in point-splitting regularization, so that the contribution in perturbation theory is

$$\left(\frac{g^2 N \, \Gamma(\omega-1)}{2^{2\omega}\pi^\omega R^{2\omega-4}} \right)^k I_{k,k} \, M_{k,k}, \quad (41)$$

where we have included the planar colour factor $(N/2)^k$. We find that

$$I_{k,k} = \frac{2\pi}{2k} \left(2 \begin{Bmatrix} \frac{1}{\epsilon} \\ -\log\delta \end{Bmatrix} \right)^k \frac{(2\pi)^{k-1}}{(k-1)!} + \text{subleading}, \quad (42)$$

and therefore, using the fact that $M_{k,k} = 2$, we have that the contribution is[12]

$$\frac{1}{k!} \left(\frac{g^2 N}{16\pi R} \right)^k \begin{Bmatrix} \left(\frac{1}{\epsilon}\right)^k \\ (-\log\delta)^k \end{Bmatrix}, \quad (43)$$

which therefore sums to

$$\exp\left(\frac{g^2 N}{16\pi R} \begin{Bmatrix} \frac{1}{\epsilon} \\ -\log\delta \end{Bmatrix} \right), \quad (44)$$

and identifying the UV cut-off $U_{\text{max.}}$ with $\exp(1/\epsilon)$ (or $1/\delta$, in the point-split case), we find that we have recovered exactly the exponential factor found from the string theory analysis (16).

We have neglected the subleading divergences, which amount to lower powers of ϵ^{-1}'s or $\log\delta$'s at each loop-order. Since these must sum-up to something less divergent than the exponential factor (44), it is natural to associate them with the prefactor $\mathcal{V}$ appearing in (16). It would be interesting to find a way to verify this idea from the string or M-theoretic perspective. These subleading divergences are of course scheme-dependent, as a redefinition of ϵ can be used to tune the subleading coefficients.

3.2   Interacting diagrams at one loop

The one-loop analysis of Wilson loops in SYM theories obtainable via dimensional reduction from $\mathcal{N} = 1$, $d = 10$ SYM appeared originally in [3]. In [45], a more general presentation was made of the same results. The two diagrams to be considered are shown in figure 3. The trivalent graph Σ_3 is built from the following function, a result

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[12]In the $k = 1$ case the integral $I_{1,1}$ has a compensatory factor of 2, accounting for the fact that $M_{1,1} = 1$.
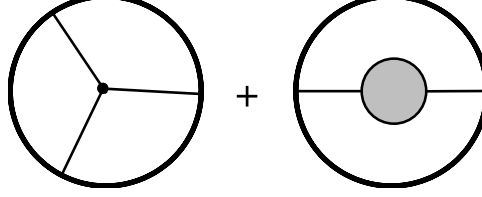
Figure 3: The one-loop, non-ladder/rainbow diagrams at $\mathcal{O}(g^4)$. One the left is Σ_3, and on the right Σ_2. Internal solid lines refer to scalar and gauge fields, while the greyed-in bubble represents the one-loop correction to the propagator.

of integrating over the position of the triple-vertex

$$
G(x_1, x_2, x_3) = \frac{\Gamma(2\omega - 3)}{2^6 \pi^{2\omega}} \int_0^1 d\alpha \, d\beta \, d\gamma \, (\alpha\beta\gamma)^{\omega-2} \delta(1 - \alpha - \beta - \gamma)
$$
$$
\times \frac{1}{\left[\alpha\beta(x_1 - x_2)^2 + \beta\gamma(x_2 - x_3)^2 + \alpha\gamma(x_1 - x_3)^2\right]^{2\omega-3}},
$$
(45)

while the one-loop-corrected propagator graph Σ_2 may be added to Σ_3 using an integration-by-parts trick [3]

$$
\Sigma_3 + \Sigma_2 = -\frac{g^4 N^2}{4} \oint d\tau_1 \, d\tau_2 \, d\tau_3 \, \epsilon(\tau_1 \, \tau_2 \, \tau_3) \left[D(\tau_1, \tau_3) \, \dot{x}_2 \cdot \partial_{x_1} G - \partial_{\tau_1} \left(D(\tau_1, \tau_3) \, G \right) \right],
$$
(46)

where $\epsilon(\tau_1 \, \tau_2 \, \tau_3)$ refers to antisymmetric path-ordering given by $+1$ for $\tau_1 > \tau_2 > \tau_3$ and totally antisymmetric in the τ_i, and where $D(\tau_1, \tau_2) = |\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2$ is the numerator of the loop-to-loop propagator (34). Plugging in the circular contour, we find

$$
\Sigma_2 + \Sigma_3 = (4 - 2\omega) \frac{g^4 N^2 \, \Gamma(2\omega - 3)}{3 \cdot 2^{2\omega+3} \pi^{2\omega} R^{4\omega-8}} \int_0^1 d\alpha \, d\beta \, d\gamma \, \delta(1 - \alpha - \beta - \gamma)
$$
$$
\times \oint d\tau_1 \, d\tau_2 \, d\tau_3 \, \epsilon(\tau_1 \, \tau_2 \, \tau_3) \frac{(\alpha\beta\gamma)^{\omega-2} \, (\sin \tau_{13} + \sin \tau_{32} + \sin \tau_{21})}{\left[\alpha\beta(1 - \cos \tau_{12}) + \beta\gamma(1 - \cos \tau_{23}) + \alpha\gamma(1 - \cos \tau_{13})\right]^{2\omega-3}}.
$$
(47)

For $d = 2\omega = 5$ this is a finite[13] integral which can be evaluated numerically. Stripping-off the factor to the left of the integral signs in (47), we find a value of -499 ± 3 using a standard Monte-Carlo integration of 10^9 steps. We find however that for $d \geq 6$, the integral is divergent.

---

[13]The convergence of this integral was discussed in [45], section 2.3. One takes one of the three Feynman parameters (e.g. γ) near zero. Integration-by-parts on one of the τ_i shows that divergences in the τ_i integration cancel. The Feynman parameter integration is then seen to be convergent only for $2\omega < 6$.

13

3.3 Generalizing to any smooth closed curve

As explained in section 2.2.3, the coefficient of the logarithmic divergence is given from string theory for a general closed contour by the following expression

$$S_{\text{rigid}} = \frac{NR_6}{2} \int d\tau_1 \left(\partial_{\tau_1}^2 x^I \right)^2. \tag{48}$$

In fact we can generalize the gauge theory analysis of the previous section by noting that the leading divergence comes from the propagators in the (k, k)-graphs pinching their ends together. Analyzing the loop-to-loop propagator in this pinched limit, one finds (again, using arclength parametrization)

$$\lim_{\tau_1 \to \tau_2} P(\tau_1, \tau_2) = \lim_{\tau_1 \to \tau_2} \frac{|\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2}{(x_1 - x_2)^{2\omega - 2}} \simeq \frac{\ddot{x}_1^2}{2\,(\tau_1 - \tau_2)^{1-\epsilon}}. \tag{49}$$

The associated path-ordered integral (we denote the circumference of the closed curve by L)[14]

$$2 \left(\frac{g^2 N}{16\pi^2} \right)^k \int_0^L d\tau_{2k} \int_0^{\tau_{2k}} d\tau_{2k-1} \cdots \int_0^{\tau_2} d\tau_1 \, P(\tau_1, \tau_2) P(\tau_3, \tau_4) \cdots P(\tau_{2k-1}, \tau_{2k}) \tag{50}$$

has a leading divergence in the pinched limit given by

$$2 \left(\frac{g^2 N}{16\pi^2} \right)^k \left(\frac{1}{2\epsilon} \right)^k \int_0^L d\tau_k \int_0^{\tau_k} d\tau_{k-1} \cdots \int_0^{\tau_2} d\tau_1 \, \ddot{x}_k^2 \cdots \ddot{x}_1^2$$
$$= \left(\frac{g^2 N}{32\pi^2\epsilon} \right)^k \frac{2}{k!} \left(\int_0^L d\tau \, \ddot{x}^2(\tau) \right)^k. \tag{51}$$

We therefore find that the leading divergence exponentiates

$$\log\langle W \rangle = \frac{g^2 N}{32\pi^2\epsilon} \int_0^L d\tau \, \ddot{x}^2 + \text{subleading}, \tag{52}$$

giving, upon using $R_6 = g^2/(8\pi^2)$ and accounting for the fact that the log of the M-theory coordinate $\tilde{U}$ shoud be identified with $1/(2\epsilon)$ (as established for the circle in sections 2.2.2 and 3.1), exactly (48). We note that a very similar exponentiation was found in the four-dimensional context in [46], where correlators of Wilson loops with operators of large R-charge were compared to string theory.

The interacting diagrams at $\mathcal{O}(g^4)$ remain finite for general smooth contours. The arguments given in footnote 13 continue to apply to well-behaved $D(\tau_1, \tau_2)$ (see (46)), i.e. for closed contours free of cusps.

Acknowledgments

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[14]The leading factor of 2 in (50) comes from $M_{k,k}$, see (36).

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